Series 9 – Solutions

Exercise 1

$$G(s) = \frac{2}{(s+1)(10s+1)}$$

To design a PI or PID regulator, we can use the Ziegler-Nichols method for which we first approximate the system in the following manner:

$$G(s) \cong \frac{Ke^{-\theta s}}{\tau s + 1}$$

In this case, we have one pole at -1, and the other pole at -0.1. The response of the system is dominated by the slower of the two modes, given by the pole at -0.1. Hence, we keep that pole in the denominator:

$$G(s) \cong \frac{2e^{-\theta s}}{10s+1}$$

We then find θ by expanding the exponential function and equating it as follows:

$$\begin{aligned}
&e^{-\theta s} = \frac{1}{s+1} \\
&\Rightarrow \frac{1}{e^{\theta s}} = \frac{1}{s+1} \\
&\Rightarrow e^{\theta s} = 1 + s \\
&\Rightarrow 1 + \theta s + \frac{\theta^2 s^2}{2} + \cdots = 1 + s \\
&\Rightarrow 1 + \theta s \ \cong 1 + s \\
&\Rightarrow \theta \cong 1
\end{aligned}$$

With this, we now have our approximation of $G(s)$:

$$G(s) \cong \frac{2e^{-s}}{10s+1}$$

We can now proceed to design our regulators

(a) PI regulator:

$$\begin{aligned}
&K_R = 0.9\frac{\tau}{\theta K} = 0.9\frac{10}{1 \cdot 2} = 4.5 \\
&\tau_I = 3.33\,\theta = 3.33
\end{aligned}$$

(b) PID regulator:

$$\begin{aligned}
&K_R = 1.2\frac{\tau}{\theta K} = 1.2\frac{10}{1 \cdot 2} = 6 \\
&\tau_I = 2\,\theta = 2 \\
&\tau_D = 0.5\,\theta = 0.5
\end{aligned}$$

Exercise 2

(a) We have a system given by

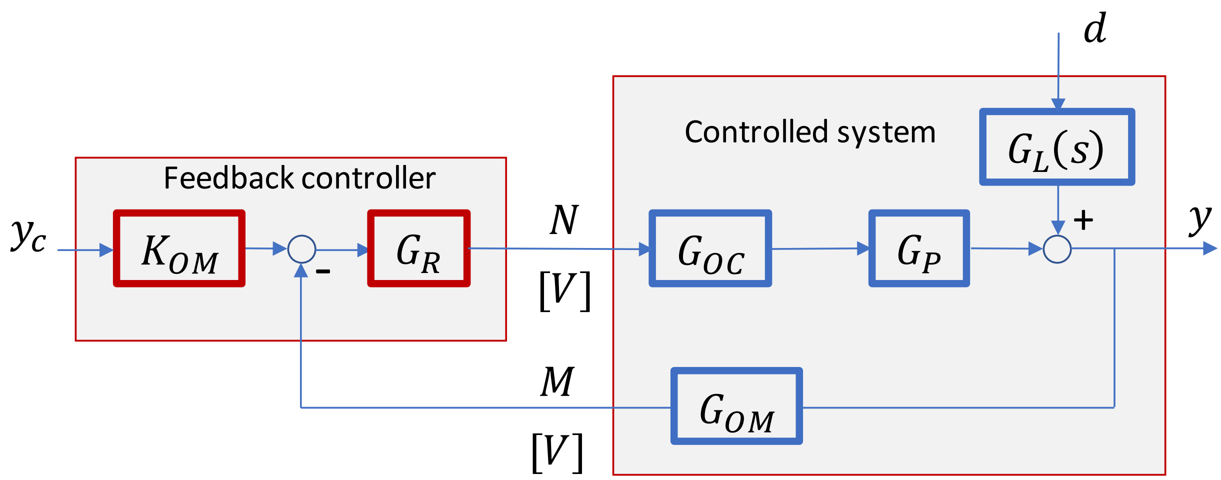
$$\frac{dy(t)}{dt} + 0.2y(t) = 0.4u(t-1)$$

The Laplace transform of this is:

$$sY(s) + 0.2Y(s) = 0.4\, e^{-s}\, U(s)$$

The transfer function that represents the system processes is:

$$G_P = \frac{Y(s)}{U(s)} = \frac{0.4}{s+0.2} \cdot e^{-s}$$



A schematic of the system. Note that in this case, there is no dynamic associated with the measuring instrument G_{OM} and the control instrument G_{OC}. Hence, they both are constants, with $G_{OM} = K_{OM}$ and $G_{OC} = K_{OC}$.

The transfer function of the measuring device, given by G_{OM} is calculated from the slope of the provided graph:

$$G_{OM} = \frac{M(s)}{Y(s)} = K_{OM} = 0.25$$

Similarly, the transfer function of the controller is given by

$$G_{OC} = \frac{U(s)}{N(s)} = K_{OC} = 10$$

The system to be controlled takes in as input N and provides as an output M. The corresponding transfer function from $N \to M$ is given by

$$G(s) = G_{OC} \cdot G_P \cdot G_{OM} = K_{OC} \cdot G_P \cdot K_{OM} = \frac{5e^{-s}}{5s+1}$$

(b) The PID controller can be designed using the Ziegler-Nichols method. Here we will have $\tau = 5$, $K = 5$, and $\theta = 1$, giving:

$$K_R = 1.2\frac{\tau}{\theta K} = 1.2\frac{5}{1 \cdot 5} = 1.2$$
$$\tau_I = 2\,\theta = 2$$
$$\tau_D = 0.5\,\theta = 0.5$$

The overall transfer function of the controller can be written as follows:

$$G_R(s) = \frac{N(s)}{E(s)} = 1.2\,(\frac{s^2 + 2s + 1}{2s})$$

(c) The overall transfer function for the closed loop system is given by the following

$$\frac{Y(s)}{Y_c(s)} = \frac{K_{OM} \cdot G_R \cdot G_{OC} \cdot G_P}{1 + G_R \cdot G_{OC} \cdot G_P \cdot G_{OM}} = \frac{K_{OM} \cdot G_R \cdot K_{OC} \cdot G_P}{1 + G_R \cdot K_{OC} \cdot G_P \cdot K_{OM}}$$

Note that the numerator corresponds to all the transfer functions that lead in a straight line from the input y_c to the output y, while the denominator is 1 + all the transfer functions inside the loop. Considering that $G(s) = K_{OM} \cdot G_P \cdot K_{OC}$, we have the following expression for $Y(s)/Y_c(s)$:

$$\frac{Y(s)}{Y_c(s)} = \frac{G_R \cdot G}{1 + G_R \cdot G}$$

$$\frac{Y(s)}{Y_c(s)} = \frac{1.2\left(\frac{s^2 + 2s + 1}{2s}\right)\left(\frac{5e^{-s}}{5s + 1}\right)}{1 + 1.2\left(\frac{s^2 + 2s + 1}{2s}\right)\left(\frac{5e^{-s}}{5s + 1}\right)}$$

$$\Rightarrow \frac{Y(s)}{Y_c(s)} = \frac{6(s^2 + 2s + 1)e^{-s}}{2s(5s + 1) + 6(s^2 + 2s + 1)e^{-s}}$$

The static gain $K_{BF} = \lim_{s \to 0} \frac{Y(s)}{Y_c(s)} = 1$

Exercise 3

The closed loop system is given by

$$G_{BF} = \frac{G_R \cdot G(s)}{1 + G_R \cdot G(s)}$$

The stability of the system is determined by the roots of the denominator. We thus analyze the characteristic equation of the system:

$$f(s) = 1 + G_R \cdot G(s) = 0$$

$$\Rightarrow 1 + \frac{2}{2s+1} \cdot \frac{K_R(\tau_I s + 1)}{\tau_I s} = 0$$

$$\Rightarrow 2\tau_I s^2 + \tau_I(1 + 2K_R)s + 2K_R = 0$$

(a) Variation in K_R

If we keep $\tau_I = 2$, and vary K_R, we get:

$$2s^2 + (1 + 2K_R)s + K_R = 0$$

The roots of this system are given by:

$$p_{1,2} = \frac{-(1 + 2K_R) \pm \sqrt{(1 + 2K_R)^2 - 8K_R}}{4}$$

$$= \frac{-(1 + 2K_R) \pm \sqrt{1 - 4K_R + 4K_R^2}}{4}$$

$$= \frac{-(1 + 2K_R) \pm (2K_R - 1)}{4}$$

This means that:

$$p_1 = -\frac{1}{2}; p_2 = -K_R$$

Here we see that one pole is always in the left hand side of the plane. The other pole is dependent on the value of K_R, with stability, and non-oscillatory behavior, guaranteed for $K_R > 0$.

(b) Variation in τ_I

To study the impact of varying τ_I on the stability of the system, we will first fix $K_R = 4$.
We then have:

$$f(s) = 2\tau_I s^2 + 9\tau_I s + 8 = 0$$

Now the roots of the system are:

$$p_{1,2} = \frac{-9\tau_I \pm \sqrt{81\tau_I^2 - 64\tau_I}}{4\tau_I}$$

$$\Rightarrow p_{1,2} = -9 \pm \sqrt{81 - \frac{64}{\tau_I}}$$

As $\tau_I \rightarrow \infty$, $p_1 \rightarrow -18, p_2 \rightarrow 0$. The slower of the two poles tends to 0 from the left hand side, but stays negative. In such a situation, the system is stable and non-oscillatory. For certain conditions, the system can be stable but oscillatory. This happens when $\tau_I < \frac{64}{81}$, leading to the term inside the square root being negative, and the formation of complex poles.